HYPER VERSION

Received: *February 25, 2000*, Accepted: *March 31, 2000*

Stable non-BPS states in F-theory

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Abstract: F-theory on K3 admits non-BPS states that are represented as string junctions extending between 7-branes. We classify the non-BPS states which are guaranteed to be stable on account of charge conservation and the existence of a region of moduli space where the 7-branes supporting the junction can be isolated from the rest of the branes. We find three possibilities; the 7-brane configurations carrying: (i) the D_1 algebra representing a D7-brane near an orientifold O7-plane, whose stable non-BPS state was identified before, (ii) the exotic affine E_1 algebra, whose stable non-BPS state seems to be genuinely non-perturbative, and, (iii) the affine E_2 algebra representing a D7-brane near a pair of O7-planes. As a byproduct of our work we construct explicitly all 7-brane configurations that can be isolated in a K3. These include non-collapsible configurations of affine type.

Keywords: Superstrings and Heterotic Strings, F-Theory, D-branes.

JHEP03(2000)036

Contents

1. Introduction and summary

Many string theories contain in their spectrum states which are non-BPS but are nevertheless stable due to the fact that they carry certain charge, and there are no other BPS or non-BPS states of lower mass carrying the same charge into which they can decay. A particular class of examples consist of a configuration where a single Dp-brane is brought close to an orientifold p-plane (Op-plane). In this case the fundamental string stretched between the D-brane and its image represents a non-BPS state. Furthermore it carries charge under a U(1) gauge field living on the D-brane, and as long as there are no other D-branes nearby, there is no BPS state of lower mass carrying this U(1) charge into which this non-BPS state can decay. Thus it will represent a stable non-BPS state. Such configurations arise in certain regions of the moduli space of the toroidal compactification of type-I string theory.

The special case of the D7-brane-O7-plane system was discussed in [1]. When non-perturbative effects are taken into account, the O7-plane splits into a pair of

7-branes [2]. We shall use the language of ref. [3, 4] to refer to the original D7-brane as an **A**-type brane and to the other two 7-branes representing the O7-plane as **B** and **C**-type branes. In this description the non-BPS state can be represented as a string junction[1] $\mathbf{j}$ with its prongs ending on the three 7-branes [3, 4].[2] In the limit when the separation between these three 7-branes is large compared to the string length scale l_s, the mass of the state $\mathbf{j}$ can be computed by adding up the masses of all the strings forming the junction [1].

Such a configuration of 7-branes arises in the special limit of F-theory compactification on elliptically fibered K3 [6] when the size of the base is large, and when the relative distances between the three 7-branes representing the D7-brane O7-plane system are much smaller than the distance between any of these three 7-branes and any of the other twenty one 7-branes. The stability of this junction $\mathbf{j}$ can be argued as follows. First of all, it is charged under a U(1) gauge field living on the **ABC** 7-brane system and carries the minimal value of charge. Second, there are no BPS states on the **ABC** brane system which carry this U(1) charge. Indeed, all states of the system carrying this U(1) charge, are non-BPS, and are represented by junctions $n\mathbf{j}$ with $n \neq 0$. In the approximation where the length of each segment of the junction is large compared to the string length scale l_s so that the mass of the junction is given by adding the masses of the strings, the states with $n = \pm 1$ are expected to be the lowest mass ones, since in this limit the state $n\mathbf{j}$ will be represented by n copies of the junction representing the state $\mathbf{j}$. Thus $\mathbf{j}$ cannot decay into a state living on the **ABC** system. In addition, since $\mathbf{j}$ carries a U(1) charge originating from the three 7-brane system, it cannot decay into a state which lives completely on the other 21 7-branes. If it were to decay, the decay products must include a junction $\mathbf{j}'$ with at least one prong extending from the **ABC** system all the way to one of the 21 other 7-branes. Since all the 21 other 7-branes are far away, the mass of $\mathbf{j}'$ is necessarily much larger than that of $\mathbf{j}$, and thus this decay is not possible due to energetic reasons. This establishes the stability of $\mathbf{j}$.

The above argument requires the relative distance between all the 7-branes to be large compared to the string length scale l_s so that the dominant contribution to the mass of the state comes from the classical mass of the junction. However, this condition can be relaxed a little. Suppose that the three special 7 branes are very close to each other, so that their separations are of the order of, or are much smaller than l_s. Then, since stringy corrections could be of order l_s^{-1} the computation of masses of the localized non-BPS junctions living on the **ABC** brane system is difficult, and one may not be able to decide which of the non-BPS junctions is the stable one against decay within this system.[3] But there will be at least one stable

[1]Throughout this paper we shall refer to both string junctions and string networks, as junctions.

[2]String junctions have been used in [5] to construct non-BPS states on the 3-brane − 7-brane system.

[3]Fortunately in the limit when the separation between the **ABC** branes are much smaller than

state, — the one with minimum mass carrying the U(1) charge. Since this state has mass of order l_s^{-1}, stringy corrections will not invalidate the conclusion of stability against decay using the faraway branes as long as the distance between the three isolated branes and every one of the twenty one other 7-branes is much larger than l_s. We will therefore have a stable non-BPS state.

In this paper we generalize this construction to other limits of F-theory compactification on K3. The basic idea will be as follows. We first consider a limit of F-theory compactification where a subset of $(24-r)$ 7-branes are far away from a set of r 7-branes. We shall call the latter a set of isolated 7-branes. In this case in analysing the stability of any state which lives solely on the isolated branes, we can forget about the existence of the other 7-branes, and study if the state can decay into other states living on the isolated set of 7-branes.[4] Then we study if this specific set of isolated branes contains a non-BPS state subject to the condition that *a*) it carries a (set of) U(1) charge(s), and *b*) there is no combination of BPS states living solely on the isolated branes which also carries the same charge quantum number(s). In this case this isolated brane configuration is guaranteed to contain a stable non-BPS state carrying charge under this U(1) gauge field(s). It is of course possible to find a set of BPS states carrying same charge quantum numbers if we include string junction configurations some of whose prongs end on the other 7-branes, but these states are too heavy, and so it is not energetically possible for the original non-BPS state to decay into such states.

This construction of course does not exhaust all possible ways of obtaining stable non-BPS states in F-theory compactification on K3. In particular one may find examples where there is a non-BPS state and a set of BPS states whose total charge quantum numbers match that of this non-BPS state, but a detailed dynamical analysis involving computation of the masses of each state shows that it is not energetically possible for a non-BPS state to decay into the set of BPS states carrying the same set of chage quantum numbers. We do not attempt to analyze these cases in this paper.

Note that if we do not consider isolated 7-brane configurations of this kind, but consider the full set of states in F-theory on K3, then for every state carrying some specific charge quantum number there is a set of BPS states carrying the same charge quantum number, and hence every state can in principle decay into a set of BPS states unless such decays are forbidden due to energetic reasons. This can be seen by noting that the full lattice of junctions for F-theory on K3 (or equivalently the Narain lattice [7] for heterotic string theory on $\mathbb{T}^2$) can be generated by a set of BPS states. Indeed, if one takes $E_8 \times E_8$ heterotic string theory on $\mathbb{T}^2$, then the $E_8 \times E_8$

l_s, the system can be described by a D7-brane O7-plane system which allows us to compute the masses, and conclude that the state carrying the minimum U(1) charge is stable.

[4] Again this argument would require the distance between the isolated branes and the other branes to be much larger than l_s.

JHEP03(2000)036

part of the lattice is generated by the root vectors of $E_8 \times E_8$ which represent massless gauge bosons and are BPS states, whereas the four dimensional lattice associated with $\mathbb{T}^2$ is generated by BPS states carrying unit winding or momentum along either of the two directions.

Our analysis proceeds in several steps. In section 2 we begin to classify 7-brane configurations which can be isolated. We find two classes of 7-brane configurations of this kind. The first class corresponds to a configuration of 7-branes where r of the 7-branes are at finite distance of each other and the other 7-branes can be pushed all the way to infinity.[5] In this case we have a non-singular background describing only these r 7-branes; and we call these *properly isolated 7-brane configurations*. The other class consists of 7-brane configurations where r of the 7-branes are within a finite distance of each other, and the other 7-branes are at a distance larger than L for some large number L. In this case, however, we cannot take the $L \to \infty$ limit and push the other branes all the way to infinity, because in this limit the string coupling constant vanishes everywhere in the region within finite distance of the isolated branes. We call these *asymptotically isolated 7-branes*. We find that the monodromy around a properly isolated brane configuration must be an elliptic or a parabolic element of the $\mathrm{SL}(2,\mathbb{Z})$ S-duality group, whereas the monodromy around an asymptotically isolated 7-brane must be a parabolic element. In particular, we see that 7-brane configurations with hyperbolic monodromies cannot be isolated.

In section 3 we look for explicit examples of isolated 7-brane configurations using the list of 7-brane configurations found in ref. [8]. In this list we find that the 7-brane configurations $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{H}_0$, $\mathbf{H}_1$, $\mathbf{H}_2$, $\widehat{\mathbf{E}}_n$ $(1 \leq n \leq 9)$, $\widehat{\widehat{\mathbf{E}}}_0$, $\widehat{\widehat{\mathbf{E}}}_1$, and $\mathbf{D}_n$ $(0 \leq n \leq 4)$ satisfy the necessary conditions for being properly isolated 7-brane configurations, and we show that these conditions are also sufficient by explicitly constructing the properly isolated 7-brane configurations of these types. The analysis is more complicated for asymptotically isolated 7-brane configurations, and we have not attempted a thorough study of all the configurations listed in ref. [8] to see which of them can be asymptotically isolated. However, we show the existence of 7-brane configurations of this kind based on $\mathbf{A}_n$ $(n \geq 0)$ and $\mathbf{D}_n$ $(n \geq 5)$ type configurations.

In section 4 we examine each of the examples of isolated 7-brane configurations found in section 3 and look for stable non-BPS states in this system. The basic idea has already been explained before: we demand that we have one or more non-BPS junctions carrying a (set of) U(1) charge(s), and that there is no combination of BPS states living on the isolated brane system carrying the same (set of) U(1) charge(s). According to our previous argument, this would guarantee that there is at least one stable non-BPS state on the isolated 7-brane system as long as the faraway branes are much farther than l_s away. We find that there are three 7-brane configurations

[5]Since in an F-theory background a constant rescaling of the metric does not destroy the solution, we can start from this solution and go to other configuraions where the distance between the r 7-branes are large or small compared to l_s.

satisfying these constraints — $\mathbf{D}_1$, $\widehat{\mathbf{E}}_2$ and $\widehat{\widehat{\mathbf{E}}}_1$. Of these $\mathbf{D}_1$ represents a D7-brane near an O7-plane, and $\widehat{\mathbf{E}}_2$ represents a D7-brane near a pair of O7-planes. The existence of possible non-BPS states in these configurations could be argued by working in the orientifold limit. On the other hand, $\widehat{\widehat{\mathbf{E}}}_1$ seems to represent a genuinely new example as it gives a non-BPS state with no simple perturbative interpretation.

In section 5 we study the possibility of obtaining non-BPS states on non-isolable 7-brane configurations. We construct several examples of 7-brane configurations containing string junctions which are stable against decay within the given 7-brane configurations. But since these 7-brane configurations are not isolable, there are other 7-branes nearby, and these junctions could be unstable against decay into string junctions which have one or more prongs ending on 7-branes outside this system.

As we were preparing to submit this paper, an interesting work by Y. Yamada and S. K. Yang appeared [9] which also gives an explicit construction of the affine exceptional brane configurations. This substantially overlaps with section 3.2.

2. Constraints on isolated configurations

Various configurations of (p,q) 7-branes were studied in refs. [10, 8]. In this section we shall analyse the conditions under which a given set of 7-branes can be considered in isolation. This issue arises because we want to consider subsets of the configuration of 24 7-branes describing F-theory on K3 [6]. We try to take an appropriate limit in the parameter space where a chosen set of 7-branes is far away from all the other 7-branes. We shall say that the chosen set of 7-branes can be isolated if it is possible to consider the limit in which the largest distance between any two members of the chosen set can be made small compared to the distance between the chosen set and any of the other 7-branes. As discussed in the introduction, only for brane configurations that can be considered in isolation we can reliably ascertain the existence of stable non-BPS states.

Our analysis of stability of non-BPS states is based on classical considerations, and thus our results are valid in the limit when the size of the S^2 base transverse to the seven branes is taken to be large compared to the string length l_s. In particular, with the help of an overall rescaling of the metric, we choose the size of the base to be sufficiently large so that the distance between the isolated branes and any of the other branes is large compared to the string length l_s. In this limit, as explained in the introduction, a stable non-BPS state on the isolated 7-brane configuration cannot be rendered unstable by the presence of the faraway branes.

Our analysis will consist of two steps. In the first step we find constraints on monodromies that can appear around an isolated configuration by requiring that all other 7-branes are at large coordinate distance away from the isolated branes. In particular, we find that hyperbolic monodromies are not allowed. Since large coordinate distance does not always correspond to large distance measured in the relevant metric, in the second step we impose the condition that when distances are measured in the appropriate metric, the isolated configuration is still far away from the remaining branes.

2.1 Constraints on monodromies for isolated configurations

A configuration of 7-branes in F-theory is described by specifying a pair of polynomials f and g in z, — the complex coordinate parametrizing the space transverse to the 7-brane. f is a polynomial of degree 8 and g is a polynomial of degree 12. We define:

$$\Delta = 4f^3 + 27g^2\,. \tag{2.1}$$

Then the dependence of the axion-dilaton modulus $\tau(z)$ on the transverse coordinate z is given by:

$$j(\tau(z)) = \frac{4\cdot(24f)^3}{\Delta}\,. \tag{2.2}$$

$j(\tau)$ blows up at the zeroes of Δ. These are the locations of the 7-branes.

We can now make more precise what we mean by isolating a set of branes. For this note that f and g are labelled by a set of 22 parameters ξ_i. We shall consider the cases where we can focus on a one dimensional subspace $\xi_i(\lambda)$, parametrized by λ. We say that we can isolate a set of branes if as we take the limit $\lambda \to 0$ the parameters $\xi_i(\lambda)$ of f and g flow in such a way that a set of r roots of Δ remain at finite points in the z plane, while the others move off to infinity. In that case for a finite but sufficiently small λ, by using scaling and translation in z, we can ensure that r of the zeroes of Δ, — which we shall associate with the location of the isolated branes, — are within the unit disk centered at the origin, and the faraway branes are outside a circle of radius L, also centered at the origin, where L is some arbitrary but fixed large number.

The above definition will describe an isolated set of r 7-branes in the sense described earlier if we can show that finite (infinite) coordinate distance in the z-plane corresponds to finite (infinite) distance measured in the metric used for computing the mass of a (p,q) string stretched along a geodesic [11, 12, 13]. This must be true for all possible values of (p,q). This constraint will give additional restrictions on the form of f and g. We shall derive these constraints in the next subsection.

Let z_i denote the positions of the isolated branes and $\widetilde{z}_i$ the position of the faraway branes. By using the additional freedom of simultaneously rescaling f and g by constants γ^2 and γ^3 respectively, we can bring Δ to the form:

$$\Delta = \prod_{i=1}^{r}(z-z_i)\prod_{i=1}^{24-r}\left(1-\frac{z}{\widetilde{z}_i}\right). \tag{2.3}$$

As mentioned before, for sufficiently small λ we have $|z_i| < 1$ for all $i = 1, \ldots, r$ and $|\widetilde{z}_i| > L$ for all $i = 1, \ldots, 24 - r$. As we travel (in the *clockwise direction*) around the set of isolated branes, the modulus τ undergoes an $\mathrm{SL}(2, \mathbb{Z})$ transformation:

$$\tau \to \frac{a\tau + b}{c\tau + d}\,. \tag{2.4}$$

We shall call the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the monodromy matrix K around the isolated set of 7-branes. Our first task will be to compute the possible monodromy matrices K around the isolated branes.[6] For computing K we can use any contour surrounding the isolated branes but not enclosing any faraway brane.

We now examine the behavior of $f(z)$ and $g(z)$. We shall call a parameter in f or g *small*, *finite* or *large*, if taking λ to zero requires the parameter to go to zero, remain bounded within a circle of finite radius in the complex plane, or grow indefinitely, respectively. We write f and g in the form

$$f(z) = F \prod_{i=1}^{d_f} (z - u_i) \prod_{i=1}^{8-d_f} \left(1 - \frac{z}{\widetilde{u}_i}\right) \tag{2.5}$$

$$g(z) = G \prod_{i=1}^{d_g} (z - v_i) \prod_{i=1}^{12-d_g} \left(1 - \frac{z}{\widetilde{v}_i}\right), \tag{2.6}$$

where we have introduced parameters $u_i, \widetilde{u}_i, v_i, \widetilde{v}_i$, entering as zeroes of f and g, and parameters F, G entering as overall coefficients. The parameters u_i $(1 \le i \le d_f)$ are assumed to be either *small* or *finite*, while the $\widetilde{u}_i$ $(1 \le i \le (8 - d_f))$ are assumed to be *large*. Similarly, v_i $(1 \le i \le d_g)$ are small or finite, and $\widetilde{v}_i$ $(1 \le i \le (12 - d_g))$ are large. The above expressions for f and g are completely general. Our analysis will require investigating the nature of the parameters F and G. Note that having used scaling to fix the distribution of branes implicit in Δ and the overall normalization of Δ, we no longer have any further scaling freedom to set F and/or G to specific values. The parameters F and G can be small, finite or large. The numbers d_f and d_g indicate the degrees of f and g respectively, when we use only the factors associated with the small and finite roots.

We shall now introduce two contours that will help in the analysis. Since we have a bounded number of small and finite roots one can define a finite length $R/2$ which is the magnitude of the largest finite root (of f, g or Δ) in the limit $\lambda \to 0$. It then follows that, for sufficiently small λ, the circle $\mathcal{C}$ of finite radius R contains all small and finite roots. Moreover, this circle is at a finite distance from all the small and finite roots of f, g and from all the isolated branes. In addition, if $\widetilde{L}$ denotes the magnitude of the smallest large root of f, g or Δ, then for sufficiently small λ we can

[6] Throughout this and the next section we shall determine the monodromy matrix only up to an $\mathrm{SL}(2, \mathbb{Z})$ conjugation, unless mentioned otherwise.

choose another circle $\mathcal{C}'$, of radius R' such that R'/R *and* $\widetilde{L}/R'$ are arbitrarily large. This circle is both far outside the finite and small roots, and far inside the large roots and the faraway branes. As mentioned before we can calculate the monodromy K using $\mathcal{C}$ or $\mathcal{C}'$, since each of them only encloses the isolated branes (note that crossing zeroes of f and g does not affect the monodromy; however in deforming $\mathcal{C}$ to $\mathcal{C}'$ we do not cross any zero of f or g).

By construction Δ takes finite values on $\mathcal{C}$ (see (2.3)). In addition, on $\mathcal{C}$, we have $f \sim F$ and $g \sim G$, by which we mean that f/F and g/G are both finite and nonzero. Since $\Delta = 4f^3 + 27g^2$, we see that both F and G cannot be small parameters. If F is large G also must be large and vice versa. Thus we need to consider the following cases separately: 1) F small, G finite 2) F finite, G small 3) F and G both finite, and 4) F and G both large.

1. F small, G finite: in this case on the contour $\mathcal{C}$, $\Delta \simeq 27g^2$, and hence from (2.1) and (2.2) we see that $j(\tau) \simeq 0$. This gives $\tau \simeq e^{2i\pi/3}$ up to an $\mathrm{SL}(2,\mathbb{Z})$ conjugation. The monodromy K around $\mathcal{C}$ must leave this value of τ fixed. This gives $K = \pm(ST)^{\pm 1}$ or $K = \pm 1$, where we define:
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.7}$$

2. F finite, G small: now $\Delta \simeq 4f^3$ along $\mathcal{C}$. This gives $j(\tau) \simeq (24)^3$, and $\tau \simeq i$ along $\mathcal{C}$. The monodromy around $\mathcal{C}$ which must leave this value of τ fixed is either $K = \pm S$ or $K = \pm 1$.

3. F and G finite: in this case we shall compute K using the contour $\mathcal{C}'$. On $\mathcal{C}'$ the functions f and g are well approximated by their overall coefficients F and G together with the factors containing the small and finite roots. There are several subcases to be considered:

   (a) $3d_f \neq 2d_g$: depending on whether $3d_f > 2d_g$ or $3d_f < 2d_g$, f or g will be the dominant contribution to Δ. In the first case $\tau \simeq i$ on $\mathcal{C}'$, and $K = \pm S$ or $K = \pm 1$. In the second case $\tau \simeq e^{2\pi i/3}$ on $\mathcal{C}'$ and $K = \pm(ST)^{\pm 1}$ or $K = \pm 1$.

   (b) $3d_f = 2d_g$. In this case there are two possibilities: $r < 3d_f$ and $r = 3d_f$. Since $\Delta = 4f^3 + 27g^2$, and f and g are approximated by polynomials of degree d_f and d_g respectively, the approximation to Δ is a polynomial of degree at most $3d_f (= 2d_q)$. Hence $r > 3d_f$ is not possible.

      i. $r < 3d_f$: in this case on $\mathcal{C}'$ we have
$$j(\tau) = \frac{4(24f)^3}{\Delta} \sim z^{3d_f - r}\,. \tag{2.8}$$

Since $j(\tau)$ is large on $\mathcal{C}'$, we can use

$$j(\tau) \sim e^{-2\pi i\tau}\,, \tag{2.9}$$

to conclude that

$$\tau \simeq -(3d_f - r)\frac{1}{2\pi i}\ln z + \text{constant}\,. \tag{2.10}$$

This implies $K = \pm T^{3d_f - r}$. Note that the power of T is positive.

ii. $r = 3d_f$: in this case f^3/Δ goes to some constant on $\mathcal{C}'$. Thus τ goes to a constant. Since f^3/Δ and g^2/Δ are both non-vanishing and finite on $\mathcal{C}'$, the constant τ is not (an SL$(2,\mathbb{Z})$ conjugate of) i, $e^{2\pi i/3}$ or $i\infty$. The monodromy along $\mathcal{C}'$ must leave fixed this constant value of τ. The only possibility is $K = \pm 1$. This case is identical to (several copies of) the D_4 case [2].

4. F and G large. In this case along $\mathcal{C}$ the function f is large but Δ is finite. Thus we can use eq. (2.9) to conclude that $\mathrm{Im}(\tau)$ must be large along $\mathcal{C}$. The monodromy around $\mathcal{C}$ must preserve this condition. This gives $K = \pm T^k$ for some integer k $(-\infty < k < \infty)$.

Note that in cases $1, 2$ and 3, the functions f, g and Δ remain well defined in the $\lambda \to 0$ limit, since they all approach finite values for finite z in the $\lambda \to 0$ limit, and each of them only has a finite number of isolated zeroes in the finite z plane. In these cases even when we set $\lambda = 0$, $\tau(z)$ is well defined in the sense that it is finite (with $\mathrm{Im}(\tau) > 0$) for finite z except at isolated points which are the locations of the 7-branes. On the other hand, in case 4, f and g do not approach a finite value for finite z as $\lambda \to 0$, since the multiplicative factors F and G blow up in this limit. Since $j(\tau) \sim f^3/\Delta$ and Δ is bounded for any finite z, for $\lambda = 0$ we will have $\mathrm{Im}(\tau) = \infty$ at every finite point in the z plane except at the zeroes of f. Since the string coupling constant is given by the inverse of the imaginary part of τ, we see that in this case the string coupling approaches zero at every finite point in the z plane except at the zeroes of f. This is a singular configuration. But since in order to get 7-brane configurations admitting stable non-BPS states we do not need to actually set $\lambda = 0$, but only need to take λ sufficiently small, even these kind of 7-brane configurations are potentially good candidates for admitting stable non-BPS states in their spectrum.

2.2 Constraints from large distance separation

Let us now study the constraints coming from the requirement that finite (large) separation in the coordinate z correspond to finite (large) separation measured in the appropriate metric used in computing the mass of a string junction living on the

7-brane system. Up to an arbitrary constant multiplicative factor, this metric used in computing the mass of a (p,q) string is given by [11]:

$$ds_{p,q} = |p+q\tau||\eta(\tau)|^2 \prod_{i=1}^{r} |z-z_i|^{-1/12}|dz|\,, \tag{2.11}$$

where z_i, as before, denote the locations of the isolated 7-branes. First let us consider the cases 1-3 discussed above. In these cases we can set $\lambda = 0$ from the beginning. Now τ is finite everywhere in the z-plane except at the locations of the 7-branes, and possibly at $z = \infty$. The metric of a (p,q)-string is known to be finite near a (p,q)-seven brane. Thus it is finite at all finite points in the z-plane, and hence finite coordinate distance will correspond to finite distance measured in the metric (2.11).[7] Thus if we can show that infinite coordinate distance corresponds to infinite distance when measured in the metric (2.11), for any (p,q) values, we would have shown that finite (large) separation in the z-coordinate system corresponds to finite (large) separation in the metric (2.11). The main issue here is whether the point $z=\infty$ is infinite or finite distance away from finite points in the z-plane. For this we note that for large $|z|$, (2.11) reduces to

$$ds_{p,q} \simeq |p+q\tau||\eta(\tau)|^2\, |z|^{-r/12}|dz|\,. \tag{2.12}$$

In cases 1, 2, 3(a) and 3(b)(ii), τ approaches a finite value as $|z| \to \infty$. From eq. (2.12) we see that in this case the point $z = \infty$ is infinite distance away for[8]

$$r \leq 12\,. \tag{2.13}$$

In case 3(b)(i) the monodromy is $\pm T^k$ with $k \equiv (3d_f - r) > 0$. In this case $\tau \to i\infty$ as $|z| \to \infty$. More precisely,

$$j(\tau) \sim e^{-2\pi i\tau} \sim z^k\,. \tag{2.14}$$

This gives, for large $|z|$,

$$\eta(\tau) \sim e^{2\pi i\tau/24} \sim z^{-k/24}\,. \tag{2.15}$$

Thus (2.12) takes the form:

$$ds_{p,q} = |p+q\tau||z|^{-\frac{r+k}{12}}|dz|\,. \tag{2.16}$$

Since $|p+q\tau|$ either approaches a finite value or grows logarithmically as $|z| \to \infty$, we see that the point $z = \infty$ is at infinite distance measured in the metric (2.16) if

$$r + k \equiv 3d_f \leq 12\,. \tag{2.17}$$

[7] The metric of a (p,q)-string may have mild divergence near a (p',q') 7-brane for $(p,q) \neq (p',q')$, but we can always choose contours which avoid such points.

[8] Similar results were obtained in ref. [11].

Finally we turn to case 4. In this case $\mathrm{Im}(\tau)$ is large for finite z, and hence

$$j(\tau) \sim e^{-2\pi i\tau} \sim F^3 \prod_{i=1}^{d_f}(z-u_i)^3 \prod_{i=1}^{r}(z-z_i)^{-1}\,. \tag{2.18}$$

This gives

$$\eta(\tau) \sim e^{2\pi i\tau/24} \sim F^{-1/8} \prod_{i=1}^{d_f}(z-u_i)^{-1/8} \prod_{i=1}^{r}(z-z_i)^{1/24}\,. \tag{2.19}$$

Hence

$$ds_{p,q} \sim |p+q\tau||F|^{-1/4} \prod_{i=1}^{d_f} |z-u_i|^{-1/4}|dz|\,. \tag{2.20}$$

Since $\tau \simeq -\frac{3}{2\pi i}\ln F$ is almost constant, we see from eq. (2.20) that the distance between two finite points in the z plane is small compared to the distance between a finite point and a point at large $|z| \ll \widetilde{L}$ provided

$$d_f \leq 4\,. \tag{2.21}$$

In the next section we shall look for 7-brane configurations satisfying all the constraints found in this section. For this analysis it will be useful to divide the possible set of isolable 7-brane configurations into two classes. Since in cases 1, 2 and 3, we can actually set $\lambda = 0$ and get a well defined function $\tau(z)$, we shall call these configurations properly isolated 7-brane configurations. On the other hand in case 4 we only get isolated 7-brane configurations for small but non-zero λ. If we try to push all the other 7-branes all the way to infinity by taking the $\lambda \to 0$ limit, $\mathrm{Im}(\tau)$ blows up for all finite z except at isolated points. We shall refer to these configurations as asymptotically isolated 7-brane configurations. As discussed earlier, both kinds of isolated 7-brane configurations are potentially relevant for finding stable non-BPS states.

3. Constructing the isolated configurations

Our analysis so far gives constraints on monodromies for isolated 7-brane configurations, but does not guarantee that given a 7-brane configuration with one of these monodromies, it can always be isolated. We can now use Table 5 of [8] to identify the brane configurations giving such monodromies. Brane configurations were classified by their monodromy matrices K into elliptic ($|\,\mathrm{Tr}(K)| < 2$), parabolic ($|\,\mathrm{Tr}(K)| = 2$) and hyperbolic ($|\,\mathrm{Tr}(K)| > 2$) type. $\pm S$ and $\pm(ST)^{\pm 1}$ are examples of elliptic monodromy, whereas $\pm T^k$ for any integer k are examples of parabolic monodromy. From our analysis we see that only 7-brane configurations with elliptic and parabolic monodromies can be possibly isolated.

Let us first consider 7-brane configurations which can be properly isolated. In this case we can have elliptic monodromies coming from cases 1, 2 or 3(a), and parabolic monodromies coming from cases 1, 2, 3(a) or 3(b) of section 2.1. In the elliptic cases equation (2.13) must be satisfied. From table 5 of ref. [8] we see that with fewer than 12 7-branes we have the configurations: $\mathbf{E}_6$, $\mathbf{E}_7$, $\mathbf{E}_8$, $\mathbf{H}_0$, $\mathbf{H}_1$, and $\mathbf{H}_2$. Parabolic cases coming from cases 1, 2, 3(a) and 3(b) have monodromy $\pm T^k$ with $k \geq 0$. Furthermore, we need to satisfy (2.17). In this class we find $\widehat{\mathbf{E}}_N$ $(9 \geq N \geq 1)$, $\widehat{\widehat{\mathbf{E}}}_1$, $\widehat{\widehat{\mathbf{E}}}_0$, and $\mathbf{D}_N$ $(4 \geq N \geq 0)$. We shall explicitly construct f and g for each of these elliptic and parabolic configurations, thereby proving that all these configurations can be properly isolated. Of these, the ones with elliptic monodromy, and the $\mathbf{D}_4$ configuration can be collapsed to a single point, but none of the other configurations with parabolic monodromy can be collapsed to a single point.

Asymptotically isolated configurations originating from case 4 in section 2.1 can have monodromy $\pm T^k$ where k is any integer. There are several configurations with monodromies of this form. For example, after excluding the properly isolated configurations, we have $\mathbf{A}_n$ $(n \geq 0)$, $\mathbf{D}_n$ $(n > 4)$, $\widehat{\mathbf{E}}_n$ $(n > 9)$, three copies of $\mathbf{D}_0$ etc.[9] But in this case a complete analysis of which of these configurations can actually be isolated is more difficult, since these configurations can be reached only as a limit. However, we do give proof of existence of a class of such configurations based on resolutions of $\mathbf{D}_N$ $(N \geq 5)$ and $\mathbf{A}_N$ $(N \geq 0)$ singularities.

We begin our analysis with the properly isolated configurations.

3.1 Elliptic cases

We now consider the configurations with elliptic monodromies having less than twelve seven-branes. These all correspond to Kodaira singularities and we will see that they can be isolated. The constraints given below on the polynomials f and g were listed in [14], which also extended the work of [15] on the construction of curves for $\mathcal{N} = 2$ supersymmetric four dimensional gauge theories with global exceptional symmetries.

$\mathbf{E}_6$: this contains eight 7-branes and has monodromy $-(ST)^{-1}$.[10] The fact that there are eight 7-branes means that for a properly isolated 7-brane configuration of this type, Δ is a polynomial of degree 8. The monodromy matrix leaves fixed the point $\tau = e^{2i\pi/3}$. Hence far away from the seven brane configuration τ must approach $e^{2i\pi/3}$, and $j(\tau)$ must vanish. This shows that f^3 must be a polynomial of degree < 8. This gives the following constraints:

$$\deg(f) \leq 2\,, \qquad \deg(g) = 4\,. \tag{3.1}$$

There are no further constraints on f and g. In order to prove that this really describes an $\mathbf{E}_6$ configuration we note that since the parameter space

[9] Two copies of $\mathbf{D}_0$ is equivalent to $\widehat{\mathbf{E}}_1$ [8].

[10] Again in this section we continue to use monodromies that are only fixed up to $\mathrm{SL}(2,\mathbb{Z})$.

is connected, it is enough to show that any one point in the parameter space describes an $\mathbf{E}_6$ configuration. If we consider the special case where $f = 0$ and $g = z^4$, then this describes an $\mathbf{E}_6$ singularity, and hence certainly represents an $\mathbf{E}_6$ type 7-brane configuration. Thus any pair of f and g satisfying (3.1) gives a properly isolated 7-brane configuration of $\mathbf{E}_6$ type. This shows that the $\mathbf{E}_6$ configuration can be isolated.

$\mathbf{E}_7$: this contains nine 7-branes, and has monodromy S. Thus Δ is a polynomial of degree 9, and τ approaches i far away from the 7-branes. The latter condition tells us that g^2/Δ must vanish sufficiently far away from the 7-branes, and hence g^2 must have degree < 9. This gives the following necessary conditions for a properly isolated 7-brane configuration of $\mathbf{E}_7$ type:

$$\deg(f) = 3\,, \qquad \deg(g) \leq 4\,. \tag{3.2}$$

Taking $f(z) = z^3$ and $g(z) = 0$ we get a collapsed configuration with $\mathbf{E}_7$ singularity. This shows that eqs. (3.2) are also sufficient for getting an $\mathbf{E}_7$ configuration.

$\mathbf{E}_8$: this has ten 7-branes and has monodromy (ST). Following the same analysis as the $\mathbf{E}_6$ case we find that the necessary and sufficient condition for having a properly isolated 7-brane configuration of this type is:

$$\deg(f) \leq 3\,, \qquad \deg(g) = 5\,. \tag{3.3}$$

$\mathbf{H}_0$: this has two 7-branes and has monodromy $(ST)^{-1}$. Following the same analysis as the $\mathbf{E}_6$ case we find,

$$\deg(f) \leq 0\,, \qquad \deg(g) = 1\,. \tag{3.4}$$

Here $\deg(f) \leq 0$ means that f can be either a constant or zero.

$\mathbf{H}_1$: this has three 7-branes and has monodromy $(S)^{-1}$. Following the same analysis as the $\mathbf{E}_7$ case we find,

$$\deg(f) = 1\,, \qquad \deg(g) \leq 1\,. \tag{3.5}$$

$\mathbf{H}_2$: this has four 7-branes and has monodromy $-(ST)$. Following the same analysis as the $\mathbf{E}_6$ case we find,

$$\deg(f) \leq 1\,, \qquad \deg(g) = 2\,. \tag{3.6}$$

This finishes all the elliptic cases. We now turn to the parabolic cases.

3.2 Properly isolated parabolic cases

Here we must deal with two series of configurations. One series carries affine exceptional algebras and the other series carries orthogonal algebras. We begin with:

3.2.1 The exceptional series $\widehat{\mathbf{E}}_n$

$\widehat{\mathbf{E}}_9$: this has twelve 7-branes and its monodromy is the identity. This corresponds to

$$\deg(f) \leq 4\,, \qquad \deg(g) \leq 6\,, \qquad \deg(\Delta) = 12\,. \tag{3.7}$$

Indeed, choosing arbitrary fourth order and sixth order polynomials for f and g respectively, it is clear that for large z we can get arbitrary constant values of τ. Only $K = \pm 1$ can leave such values invariant. On the other hand, it follows from the arguments of ref. [8], section 4 that twelve 7-branes cannot produce $K = -1$. We must therefore have $K = 1$. This configuration arises from two copies of the $\mathbf{D}_4$ case discussed in [2].

If we define the coefficients f_k, g_k and d_k through the relations:

$$\begin{aligned}
\overline{f}(z) &\equiv -(4)^{1/3} f(z) = \sum_{k=0}^{4} f_k z^k \\
\overline{g}(z) &\equiv (27)^{1/2} g(z) = \sum_{k=0}^{6} g_k z^k \\
\Delta &\equiv -\overline{f}^{\,3} + \overline{g}^{\,2} = \sum_{k=0}^{12} d_k z^k\,,
\end{aligned} \tag{3.8}$$

then, in order to get $\widehat{\mathbf{E}}_9$ one must have $d_{12} \neq 0$, which requires $g_6^2 - f_4^3$ to be non-zero.

$\widehat{\mathbf{E}}_n$ ($2 \leq n \leq 8$), $\widehat{\widehat{\mathbf{E}}}_1$, $\widehat{\widehat{\mathbf{E}}}_0$: the $\widehat{\mathbf{E}}_n$ configuration has $(n+3)$ seven branes, and has monodromy T^{9-n}. Thus for a properly isolated brane configuration of this kind, Δ is a polynomial of degree $(n+3)$, and $j(\tau)$ for large but finite z behaves as

$$j(\tau) \sim z^{9-n}\,. \tag{3.9}$$

Using (2.1) and (2.2) we see that $f^3 \sim z^{12}$, and $g^2 \sim z^{12}$ for large $|z|$. Thus f is a polynomial of degree 4 and g is a polynomial of degree 6. Let us introduce the coefficients of expansion f_k, g_k through the relations (3.8). Using the freedom of shifting z, and the freedom of scaling f and g by γ^2 and γ^3 respectively for any complex number γ, we set

$$f_0 = 0\,, \qquad f_4 = 1\,. \tag{3.10}$$

This still leaves a residual rescaling freedom where we scale g by -1 and leave f unchanged; we shall use this later. We are also left with the freedom of scaling z by a complex number K together with a compensating scaling $f \to K^{-4} f$, $g \to K^{-6} g$ so as to preserve the $f_4 = 1$ condition. We shall also make use of this later.

In order to get the $\widehat{\mathbf{E}}_n$ configuration, we need to ensure that the coefficients d_k defined in eq. (3.8) vanish for $k \geq (n+4)$, and that d_{n+3} does not vanish. We shall begin by describing the solution for $\widehat{\mathbf{E}}_2$. First of all, requiring the coefficient d_{12} to vanish we get $g_6 = \pm 1$. We can now use the residual scaling freedom $g \to -g$, $f \to f$ to set,

$$g_6 = 1 \equiv \widehat{g}_6 \,. \tag{3.11}$$

Now by equating the coefficients of $d_{11}, \ldots, d_6$ to 0, we get:[11]

$$d_{11} = 0: \qquad g_5 = \frac{3}{2} f_3 \equiv \widehat{g}_5 \,, \tag{3.12}$$

$$\begin{aligned} d_{10} = 0: \qquad g_4 &= \frac{1}{2}(3f_2 + 3f_3^2 - g_5^2) \\ &= \frac{3}{2} f_2 + \frac{3}{8} f_3^2 \equiv \widehat{g}_4 \,. \end{aligned} \tag{3.13}$$

$$\begin{aligned} d_9 = 0: \qquad g_3 &= \frac{1}{2}(3f_1 + 6f_3 f_2 + f_3^3 - 2g_4 g_5) \\ &= \frac{3}{2} f_1 + \frac{3}{4} f_2 f_3 - \frac{1}{16} f_3^3 \equiv \widehat{g}_3 \,. \end{aligned} \tag{3.14}$$

$$\begin{aligned} d_8 = 0: \qquad g_2 &= \frac{1}{2}(6f_1 f_3 + 3f_2^2 + 3f_3^2 f_2 - 2g_3 g_5 - g_4^2) \\ &= \frac{3}{8} f_2^2 - \frac{3}{16} f_2 f_3^2 + \frac{3}{4} f_1 f_3 + \frac{3}{128} f_3^4 \equiv \widehat{g}_2 \end{aligned} \tag{3.15}$$

$$\begin{aligned} d_7 = 0: \qquad g_1 &= \frac{1}{2}(6f_1 f_2 + 3f_1 f_3^2 + 3f_3 f_2^2 - 2g_2 g_5 - 2g_3 g_4) \\ &= \frac{3}{4} f_1 f_2 - \frac{3}{16} f_2^2 f_3 + \frac{3}{32} f_2 f_3^3 - \frac{3}{16} f_1 f_3^2 - \frac{3}{256} f_3^5 \equiv \widehat{g}_1 \,. \end{aligned} \tag{3.16}$$

$$\begin{aligned} d_6 = 0: \qquad g_0 &= \frac{1}{2}(3f_1^2 + 6f_1 f_2 f_3 + f_2^3 - 2g_1 g_5 - 2g_2 g_4 - g_3^2) \\ &= \frac{3}{8} f_1^2 - \frac{3}{8} f_1 f_2 f_3 - \frac{1}{16} f_2^3 + \frac{9}{64} f_2^2 f_3^2 - \frac{15}{256} f_2 f_3^4 + \\ &\quad + \frac{3}{32} f_1 f_3^3 + \frac{7}{1024} f_3^6 \equiv \widehat{g}_0 \,. \end{aligned} \tag{3.17}$$

This determines the parameters g_i in terms of three independent parameters f_1, f_2, f_3. In order to have an $\widehat{\mathbf{E}}_2$ configuration we must also require d_5 to be non-zero. A straightforward computation gives:

$$d_5 = \frac{3}{1024}(8f_1 - 4f_2 f_3 + f_3^3)(16f_2^2 + 16f_1 f_3 - 16f_2 f_3^2 + 3f_3^4) \,. \tag{3.18}$$

[11] This calculation is straightforward but complicated, and has been done with the help of the algebraic manipulator programme MAPLE.

Thus the most general properly isolated $\widehat{\mathbf{E}}_2$ configuration is labelled by three parameters $f_1, \ldots, f_3$ satisfying the inequality $d_5 \neq 0$. Of these three parameters one is redundant due to the freedom of scaling of z. Using this freedom, we can require the d_5 given in (3.18) to be equal to $(3/1024)$. This gives a two parameter family of $\widehat{\mathbf{E}}_2$ configurations. In summary

$$\begin{aligned}\widehat{\mathbf{E}}_2 : \overline{f}(z) &= f_1 z + f_2 z^2 + f_3 z^3 + z^4 \,, \\ \overline{g}(z) &= \sum_{k=0}^{6} \hat{g}_k z^k \,, \\ (8f_1 - 4f_2 f_3 + f_3^3)&(16f_2^2 + 16 f_1 f_3 - 16 f_2 f_3^2 + 3f_3^4) = 1 \,. \end{aligned} \tag{3.19}$$

The construction of properly isolated $\widehat{\mathbf{E}}_n$ configurations with $2 < n < 9$ follows trivially. In this case we require d_k to vanish for $k \geq (n+4)$, and d_{n+3} to be non-zero. Thus we need to satisfy the first $(9-n)$ of the equations (3.11)–(3.17), and also require that the left hand side minus the right hand side of the $(10-n)$th equation $(g_{n-3} - \widehat{g}_{n-3})$ be non-zero, which we can set equal to $(+1)$ by using the freedom of rescaling z. Thus the general solution is parametrized by $(n+1)$ parameters $f_1, \ldots, f_3, g_0, \ldots, g_{n-3}$ subject to one 'gauge fixing condition', which determines g_{n-3} in terms of the other parameters. Using the gauge fixing condition $g_{n-3} - \hat{g}_{n-3} = 1$, the explicit solution is given by

$$\begin{aligned}\widehat{\mathbf{E}}_{9>n>2} : \quad \overline{f}(z) &= f_1 z + f_2 z^2 + f_3 z^3 + z^4 \,, \\ \overline{g}(z) &= \hat{g}_6 \, z^6 + \cdots + \hat{g}_{n-2} z^{n-2} + (1 + \hat{g}_{n-3}) z^{n-3} + \sum_{k=0}^{n-4} g_k z^k \,, \end{aligned} \tag{3.20}$$

where $\hat{g}_6 = 1$, and the other $\hat{g}_n$ are given in (3.12)–(3.17). $g_0, \ldots, g_{n-4}$ are arbitrary.

Let us now turn to the case of $\widehat{\mathbf{E}}_1$. In this case we need to satisfy eqs. (3.11)–(3.17), together with $d_5 = 0$. It follows from (3.18) that we need

$$(8f_1 - 4f_2 f_3 + f_3^3)(16f_2^2 + 16 f_1 f_3 - 16 f_2 f_3^2 + 3f_3^4) = 0 \,. \tag{3.21}$$

Note that this equation contains two factors. Furthermore, it is straightforward to see that if we require both factors to vanish simultaneously, then Δ vanishes identically, and hence we have an unphysical solution.

Thus it appears that in the parameter space labelled by f_1, f_2, and f_3, there are two physically disconnected regions which give properly isolated 7-brane configurations with the same number of 7-branes and the same monodromy as the $\widehat{\mathbf{E}}_1$ configuration:

$$(8f_1 - 4f_2 f_3 + f_3^3) = 0 \,, \tag{3.22}$$

or

$$(16f_2^2 + 16 f_1 f_3 - 16 f_2 f_3^2 + 3f_3^4) = 0 \,. \tag{3.23}$$

It turns out that among the configurations of 7-branes listed in table 5 of [8] there is precisely one more configuration with the same monodromy and the same number of 7-branes as the $\widehat{\mathbf{E}}_1$ configuration, namely the $\widehat{\widehat{\mathbf{E}}}_1$ configuration. Thus we expect to identify one of the branches of (3.21) with $\widehat{\widehat{\mathbf{E}}}_1$, and the other branch with $\widehat{\mathbf{E}}_1$. Let us begin with the first branch, given by equation (3.22) which we can use to solve for f_1 as

$$f_1 = -\frac{1}{8} f_3^3 + \frac{1}{2} f_2 f_3 \,. \tag{3.24}$$

With this value of f_1 we can now find

$$d_4 = -\frac{3}{16384} (4f_2 - f_3^2)^4 \,. \tag{3.25}$$

We have two parameters f_2 and f_3. It is convenient at this stage to introduce a new parameter s through the relation $f_2 = \frac{1}{4} s f_3^2$, and then set $f_3 = 4$ as a gauge condition.[12] We then have $d_4 = -12(s-1)^4$, and therefore we should get the desired configuration when $s \neq 1$. At this stage we can write every coefficient in terms of s. As we will explain shortly, this is $\widehat{\mathbf{E}}_1$. We then have

$$\begin{aligned} \widehat{\mathbf{E}}_1 : \quad \overline{f}(z) &= z^4 + 4z^3 + 4s\, z^2 + 8(s-1)\, z \,, \qquad s \neq 1 \\ \overline{g}(z) &= z^6 + 6z^5 + 6(s+1)z^4 + (24s-16)\, z^3 + \\ &\quad + 6(s+3)(s-1)z^2 + 12(s-1)^2\, z - 4(s-1)^3 \,. \end{aligned} \tag{3.26}$$

From this one finds

$$\Delta(\widehat{\mathbf{E}}_1) = 4(s-1)^4 \left(-3z^4 - 12z^3 - 12s\, z^2 - 24(s-1)z + 4(s-1)^2\right). \tag{3.27}$$

Let us now confirm that this is $\widehat{\mathbf{E}}_1$. To this end we recall that the $\widehat{\mathbf{E}}_1$ brane configuration is **BCBC** and either the **B** or the **C** branes can be brought together to define an $\mathbf{A}_1$ singularity. Indeed, we found that letting

$$z = -1 + y \,, \qquad s = -\frac{1}{2} - \sqrt{3} \tag{3.28}$$

equations (3.26) and (3.27) become

$$\begin{aligned} \overline{f}(y) &= (7 + 4\sqrt{3}) - (8 + 4\sqrt{3})y^2 + y^4 \\ \overline{g}(y) &= (26 + 15\sqrt{3}) + \left(30\sqrt{3} + \frac{105}{2}\right) y^2 - (12 + 6\sqrt{3})\, y^4 + y^6 \\ \Delta(y) &= -\frac{27}{4}(97 + 56\sqrt{3})\, y^2\, (y^2 - 4\sqrt{3} - 8) \,. \end{aligned} \tag{3.29}$$

This is an $\mathbf{A}_1$ singularity at $y = 0$; indeed, at this point we have $\mathrm{ord}(f) = \mathrm{ord}(g) = 0$, and $\mathrm{ord}(\Delta) = 2$.

[12] Note that this gauge condition is valid for all (f_2, f_3) as long as $f_3 \neq 0$, i.e. $s \neq \infty$. If f_3 vanishes, then we need to choose a different gauge.

This confirms that we are dealing with $\widehat{\mathbf{E}}_1$. We can perform another check. It should not be possible to decouple a brane in this configuration, since removing any single brane from $\widehat{\mathbf{E}}_1$ will leave a configuration with hyperbolic monodromy [8] and such configuration (by our earlier arguments) cannot be isolated. Indeed, to make the coefficient of z^4 in Δ vanish, we must take $s = 1$, but this makes Δ vanish identically, and therefore this is not a physical solution.[13]

We now begin the exploration of the second branch, indicated in (3.23). In here we must set:

$$f_1 = -\frac{f_2^2}{f_3} + f_2 f_3 - \frac{3}{16} f_3^3 \,. \tag{3.30}$$

With this condition, we now examine the resulting value of d_4 which turns out to be

$$d_4 = \frac{3}{16384} \frac{(8f_2 - 3f_3^2)(4f_2 - f_3^2)^4}{f_3^2} \,. \tag{3.31}$$

We have two parameters f_2 and f_3. It is convenient at this stage to relate them via another parameter. We put $f_2 = \frac{1}{4} s f_3^2$, and then to set $f_3 = 4$ as a gauge condition.[14] We then find

$$\begin{aligned} \widehat{\widehat{\mathbf{E}}}_1 : \quad \overline{f}(z) &= z^4 + 4z^3 + 4sz^2 - 4(s-1)(s-3)z \,, \qquad s \neq 1 \\ \overline{g}(z) &= z^6 + 6z^5 + 6(1+s)z^4 + (-22 + 36s - 6s^2)z^3 - 6(s-1)(s-5)z^2 - \\ &\quad - 12(s-2)(s-1)^2 \, z + 2(3s-5)(s-1)^3 \,. \end{aligned} \tag{3.32}$$

and one can confirm that

$$\begin{aligned} \Delta(\widehat{\widehat{\mathbf{E}}}_1) = (s-1)^4 \Big(& 12(2s-3)\, z^4 - 8(s^2 - 14s + 19)\, z^3 + 24(3s^2 - 4s - 1)\, z^2 - \\ & - 48(3s-5)(s-2)(s-1)\, z + 4(3s-5)^2(s-1)^2 \Big) \,. \end{aligned} \tag{3.33}$$

This is the one-parameter presentation of $\widehat{\widehat{\mathbf{E}}}_1$. To confirm this end we recall that $\widehat{\widehat{\mathbf{E}}}_1$ is described as $\mathbf{AX}_{[2,-1]}\mathbf{CX}_{[4,1]}$, and the $\mathbf{A}$ and $\mathbf{C}$ branes, for example, can be brought together at $z = 0$. For this one must have $f \sim z$ and $g \sim z$. Since $\overline{f} \sim z$ in the above, we must see if it is possible to set to zero the z-independent term in $\overline{g}$. Indeed, we see two possibilities. The first one, $s = 1$ is ruled out, since then Δ vanishes identically. On the other hand we can take $s = 5/3$. This gives

$$\overline{f}(z) = z^4 + 4z^3 + \frac{20}{3} z^2 + \frac{32}{9} z$$

[13] We also need to make sure that the special point $s = \infty$, where our gauge choice breaks down, does not correspond to a decoupled brane configuration. To see this we go back to eq. (3.25) and set $f_3 = 0$. Requiring d_4 to vanish will now require f_2 to vanish. This, in turn, makes f_1 and all the coefficients g_k for $0 \leq k \leq 5$ to vanish. Thus we get $\bar{f} = z^4$, and $\bar{g} = z^6$. This makes $\Delta = \bar{g}^2 - \bar{f}^3$ vanish identically.

[14] Again, this gauge condition breaks down if $f_3 = 0$.

$$\overline{g}(z) = z^6 + 6z^5 + 16z^4 + \frac{64}{3}z^3 + \frac{40}{3}z^2 + \frac{16}{9}z$$
$$\Delta = \frac{64}{81}\, z^2\Big(z^2 + \frac{28}{9}z + 4\Big)\,. \tag{3.34}$$

This is indeed $\widehat{\widehat{\mathbf{E}}}_1$ with an $\mathbf{H}_0$ singularity at $z = 0$.

Finally, we can identify the configuration $\widehat{\widehat{\mathbf{E}}}_0$ which has three 7-branes, by decoupling a brane from $\widehat{\widehat{\mathbf{E}}}_1$. It is clear that we must set to zero the coefficient of z^4 in Δ as given in (3.33). Taking $s = 1$ is clearly illegal, so we must take $s = 3/2$. In this case we find

$$\overline{f}(z) = z^4 + 4z^3 + 6z^2 + 3z$$
$$\overline{g}(z) = z^6 + 6z^5 + 15z^4 + \frac{37}{2}z^3 + \frac{21}{2}z^2 + \frac{3}{2}z - \frac{1}{8}\,. \tag{3.35}$$

This can be simplified by letting $z \to z - 1$. One then obtains:

$$\widehat{\widehat{\mathbf{E}}}_0 : \qquad \overline{f}(z) = z^4 - z$$
$$\overline{g}(z) = z^6 - \frac{3}{2}z^3 + \frac{3}{8}$$
$$\Delta = -\frac{1}{8}z^3 + \frac{9}{64}\,. \tag{3.36}$$

The same shift $z \to z - 1$ would also simplify somewhat the presentations of $\widehat{\mathbf{E}}_1$ and $\widehat{\widehat{\mathbf{E}}}_1$ given earlier. This concludes our proof that all the $\widehat{\mathbf{E}}_n$ $(1 \le n \le 9)$ and $\widehat{\widehat{\mathbf{E}}}_n$ $(n = 0, 1)$ configurations can be properly isolated.

3.2.2 The orthogonal series $\mathbf{D}_n$ $(0 \le n \le 4)$

The $\mathbf{D}_n$ configuration has monodromy $-T^{4-n}$ and has $(n + 2)$ 7-branes. The appropriate f and g in these cases coincide with the corresponding functions found in ref. [16] for $\mathcal{N} = 2$ supersymmetric SU(2) gauge theories with n hypermultiplets in the fundamental representation [2]. However for completeness we shall construct these functions explicitly here, as it does not require any extra effort.

Proceeding in the same way as in the $\widehat{\mathbf{E}}_n$ case, we conclude that the $\mathbf{D}_n$ configuration for $n \le 4$ is described by polynomials f and g of degree 2 and 3 respectively, subject to the condition that Δ is a polynomial of degree $(n + 2)$. We introduce the coefficients f_k, g_k and d_k through the equations:

$$\bar{f}(z) \equiv -(4)^{1/3} f(z) = \sum_{k=0}^{2} f_k z^k$$
$$\bar{g}(z) \equiv (27)^{1/2} g(z) = \sum_{k=0}^{3} g_k z^k$$
$$\Delta \equiv -\bar{f}^3 + \bar{g}^2 = \sum_{k=0}^{6} d_k z^k\,. \tag{3.37}$$

Then for $\mathbf{D}_4$ the only requirement on the coefficients is that $g_3^2 \neq f_2^3$. In other words:

$$\deg(f) \leq 2\,, \qquad \deg(g) \leq 3\,, \qquad \deg(\Delta) = 6\,. \tag{3.38}$$

In order to get a $\mathbf{D}_n$ configuration for $n \leq 3$, d_k must vanish for $k \geq (n+3)$, and d_{n+2} should be non-zero. This gives constraints on the coefficients f_n and g_n. As in the $\widehat{\mathbf{E}}_n$ case, by using the freedom of shifting z and rescaling f and g we set

$$f_0 = 0\,, \qquad f_2 = 1\,. \tag{3.39}$$

Instead of discussing the case of each of the $\mathbf{D}_n$'s separately, it is most convenient to start with $\mathbf{D}_0$. By equating the coefficients of d_6, d_5, d_4 and d_3 to zero we find the following constraints:

$$d_6 = 0: \quad g_3 = 1 \equiv \widehat{g}_3 \tag{3.40}$$

$$d_5 = 0: \quad g_2 = \frac{3}{2} f_1 \equiv \widehat{g}_2 \tag{3.41}$$

$$d_4 = 0: \quad g_1 = \frac{1}{2}(3f_1^2 - g_2^2) = \frac{3}{8} f_1^2 \equiv \widehat{g}_1 \tag{3.42}$$

$$d_3 = 0: \quad g_0 = \frac{1}{2}(f_1^3 - 2g_1 g_2) = -\frac{1}{16} f_1^3 = \widehat{g}_0\,. \tag{3.43}$$

These equations determine the coefficients g_k in terms of the single parameter f_1. Using eqs. (3.39)–(3.43) we get $d_2 = -\frac{3}{64} f_1^4$. This term must not vanish, and therefore $f_1 \neq 0$. We can use z-scaling together with compensating f and g scalings (as before) to fix $f_1 = 4$, while preserving $f_4 = 1$. We then have for $\mathbf{D}_0$

$$\begin{aligned} \mathbf{D}_0: \quad \overline{f}(z) &= z^2 + 4z \\ \overline{g}(z) &= z^3 + 6z^2 + 6z - 4 \\ \Delta(z) &= -12z^2 - 48z + 16\,. \end{aligned} \tag{3.44}$$

The solution for $\mathbf{D}_n$ for all other $n \leq 3$ is now easily obtained. For this the coefficients g_n need to satisfy the first $(4-n)$ equations in eq. (3.40)–(3.43), and should not satisfy the $(5-n)$th of these equations. This determines the parameters $g_n, g_{n+1}, \ldots, g_3$ in terms of f_1, and gives a strict inequality for g_{n-1}. As in the $\widehat{\mathbf{E}}_n$ case, we can use the freedom of scaling z to ensure that the difference between the left and the right hand side of the $(5-n)$th equation is 1. This gauge fixing condition determines g_{n-1}. Thus the general solution is parametrized by n parameters $f_1, g_0, \ldots, g_{n-2}$. Explicitly, the answer is:

$$\begin{aligned} \widehat{\mathbf{D}}_{n>0}: \quad \overline{f}(z) &= z^2 + f_1 z \\ \overline{g}(z) &= \hat{g}_3\, z^3 + \cdots + \hat{g}_n z^n + (1 + \hat{g}_{n-1}) z^{n-1} + \sum_{k=0}^{n-2} g_k z^k \end{aligned} \tag{3.45}$$

where $\hat{g}_3 = 1$, the other $\hat{g}_n$ are given in (3.41)–(3.43), and $g_0, \ldots, g_{n-2}$ are arbitrary.

This finishes explicit construction of all the properly isolated 7-brane configurations.

3.3 Examples of asymptotically isolated 7-brane configurations

We now turn to the asymptotically isolated 7-brane configurations. As stated earlier, we shall not attempt to completely classify or to give explicit constructions of all such 7-brane configurations. But we shall consider two examples.

$\mathbf{D}_n$ $(n > 4)$: in this case the monodromy, $-T^{4-n}$, is proportional to a negative power of T. From our analysis in the last section we see that the only way such a monodromy can be obtained is in case 4, where the overall coefficients in the functions f and g blow up in the $\lambda \to 0$ limit. Thus these configurations cannot be properly isolated. Indeed, for these configurations the string coupling $(\mathrm{Im}(\tau))^{-1}$ grows at large distance, and in order to prevent it from blowing up at a finite value of z (which would represent other 7-branes), the string coupling at finite points in the z plane must be made smaller and smaller as λ approaches zero. We shall not attempt to give an explicit construction of these configurations here. However, the existence of such configurations can be proved as follows. We start with a collapsed $\mathbf{D}_n$ configuration at $z = 0$ which is known to exist for $n \geq 4$, and resolve the singularity slightly so that the branes in the $\mathbf{D}_n$ configuration are located at $z \sim \epsilon$ for some small number ϵ. Now we can rescale z by $(1/\epsilon)$ to put these branes at finite values of z. This takes all the other branes to large values of z.

In order to verify that these configurations satisfy condition (2.21), we note that at a $\mathbf{D}_n$ singularity $f(z)$ has a double zero. Thus, after resolving the singularity and rescaling, $f(z)$ has two zeroes at finite z. This gives $d_f = 2$, which satisfies the inequality (2.21).

This proves the existence of asymptotically isolated $\mathbf{D}_n$ configurations for $n > 4$.

$\mathbf{A}_n$: this configuration has $(n+1)$ 7-branes and has monodromy $T^{-(n+1)}$. Thus it also belongs to the class of 7-branes which can only be asymptotically isolated. Again we shall not discuss explicit construction of these configurations. The existence of such configurations can be proved in the same way as the $\mathbf{D}_n$ case for $n > 4$ by resolving a configuration with $\mathbf{A}_n$ singularity, followed by a scaling of z. For an $\mathbf{A}_n$ singularity $f(z)$ has no zeroes at the location of the singularity. Thus after resolution of the singularity and appropriate rescaling, there will be no zero of f at a finite value of z. This gives $d_f = 0$, which satisfies (2.21).

4. Brane configurations with stable non-BPS states

In the present section we investigate which 7-brane configurations support stable non-BPS states. Such states take the form of string junctions extending between the seven branes.[15] We will focus on brane configurations that can be isolated in the sense discussed in the previous sections. Unless the brane configuration can be isolated we cannot reliably ascertain the stability of the candidate states. In a later section we will discuss some aspects of non-isolable configurations and their potentially stable non-BPS states.

Once we focus on a particular brane configuration, we only examine string junctions joining 7-branes of that configuration. We call these *localized junctions*, since they do not carry away charge to some remote 3-brane or to another set of 7-branes. We now claim that a string junction $\mathbf{J}$ on such 7-brane configuration corresponds to a *possibly stable non-BPS state* if:

(i) The associated homology cycle $\mathbf{J}$ satisfies $\mathbf{J}^2 < -2$.

(ii) $\mathbf{J} \neq \sum_i n_i \mathbf{j}_i$, where $\mathbf{j}_i$ are homology cycles satisfying $\mathbf{j}_i^2 \geq -2$ and n_i are arbitrary integers.

Let us first examine the first condition. Recall that in F-theory on an elliptically fibered K3 over base S^2, a string junction joining type-IIB seven-branes on S^2 can be associated to a two cycle in K3. This cycle, being boundaryless, corresponds to an element of the second homology class of K3. We use the symbol $\mathbf{J}$ to denote interchangeably the junction and the associated homology cycle. We also denote by $\mathbf{J}^2$ the self-intersection number of the cycle. It is well-known that in K3 any cycle with $\mathbf{J}^2 = 2g - 2$ $(g \geq 0)$ has a holomorphic representative of genus g. That representative defines a BPS junction. On the other hand, when $\mathbf{J}^2 < -2$ the cycle has no holomorphic representative, and the associated junction is never BPS. Thus the first condition guarantees that the state is not BPS.[16]

Let us now consider the second condition. Suppose $\mathbf{J} = \sum_i n_i \mathbf{j}_i$, where $\mathbf{j}_i$ are homology cycles satisfying $\mathbf{j}_i^2 \geq -2$ and n_i are some integers. The equality of homology cycles implies that whatever charges $\mathbf{J}$ carries they are also carried by the total set of states associated to the right hand side. Since all states in the right hand side are BPS ($\mathbf{j}_i^2 \geq -2$) the decay of $\mathbf{J}$ into stable BPS states cannot be ruled out by charge conservation. Therefore, condition (ii) ensures that the state cannot decay into stable BPS states. Of course, even if condition (ii) is not satisfied, the non-BPS state will be stable if its mass is lower than the sum of the masses of the possible product states. But this requires a detailed study of the masses of various states. We do not attempt to carry out such analysis here.

[15] When we refer to a junction corresponding to a specific vector in the junction lattice, it corresponds to the minimal mass configuration among a whole set of junctions which can be continuously deformed to each other by manipulations of the form discussed in [3] and [4].

[16] This can also be seen in the dual heterotic string theory on $\mathbb{T}^2$ as follows. In the heterotic description $\mathbf{J}$ corresponds to a vector on the Narain lattice [7] and $\mathbf{J}^2$ corresponds to its squared norm. Since in the heterotic string theory there are no BPS states with $\mathbf{J}^2 < -2$, we see string junctions with $\mathbf{J}^2 < -2$ cannot be BPS.

In general, a brane configuration will admit many or infinite number of possibly stable non-BPS states, namely states satisfying conditions (i) and (ii). Such states may decay into each other, but there will be at least one state — the lightest of the possibly non-BPS stable states — or perhaps more that will be *genuinely stable non-BPS state(s)*. Which particular states are stable, and the number of such states, may vary as we change the parameters labelling the isolated 7-brane configuration.

Given a 7-brane configuration with $(N+2)$ branes, the fact that the junction does not carry away charge imposes two conditions (unless all branes are mutually local) — one corresponding to the D-string charge and another corresponding to the fundamental string charge — and therefore the set of localized junctions is spanned by N linearly independent junctions. If one identifies a semisimple algebra of rank N on this brane configuration there cannot be stable non-BPS states. Indeed, having identified a rank N semisimple algebra means having identified a set of N linearly independent localized BPS junctions representing the simple roots of the algebra [10, 8]. This is therefore a basis for the set of all localized junctions, and therefore any junction can be written as some integral linear combination of junctions that are BPS, in violation of condition (ii). What one needs is u(1) factors in the algebra carried by the branes. Such factors arise when holomorphic junctions do not span the lattice of localized junctions.

We shall first examine the basic realizations of the (extended) $\mathbf{A}$, $\mathbf{D}$, $\mathbf{E}$, $\widehat{\mathbf{E}}$ and $\mathbf{H}$ series and find the cases that can give rise to possibly stable non-BPS states. We follow the convention of refs. [10, 8] of denoting by $\mathbf{X}_{p,q}$ the (p,q) 7-brane with monodromy $K = \left(\begin{smallmatrix} 1+pq & -p^2 \\ -q^2 & 1-pq \end{smallmatrix}\right)$, and define special 7-branes $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ as $\mathbf{A} = \mathbf{X}_{[1,0]}$, $\mathbf{B} = \mathbf{X}_{[1,-1]}$ and $\mathbf{C} = \mathbf{X}_{[1,1]}$. The monodromy of a brane configuration containing a product of $\mathbf{X}_{p,q}$'s is obtained by multiplying the individual monodromy matrices in opposite order.

- The $\mathbf{A}_N$ series ($N \geq 1$). These configurations are special in that all branes are mutually local; the configuration is produced by $(N+1)$ $\mathbf{A}$ branes. With just one charge conservation condition corresponding to the fundamental string charge, localized junctions are spanned by N basis elements. The N junctions joining $\mathbf{A}_i$ to $\mathbf{A}_{i+1}$ represent the su$(n+1)$ roots and span the lattice of localized junctions. Thus (ii) cannot be satisfied.

- The $\mathbf{D}_N$ series ($N \geq 0$). These brane configurations are $\mathbf{D}_N = \mathbf{A}^N\mathbf{BC}$. For the case $\mathbf{D}_0$, which only has two branes, there are no localized junctions. The configuration $\mathbf{D}_1$ carries a u(1) algebra only and thus has a candidate non-BPS state. On the other hand for $\mathbf{D}_{N\geq 2}$ the algebra is semisimple and therefore there are no possibly stable non-BPS states.

- The $\mathbf{H}_N$ ($N \geq 0$). Only $\mathbf{H}_{N\leq 3}$ can be isolated. ($\mathbf{H}_3 = \mathbf{D}_3$). This series is realized as $\mathbf{H}_N = \mathbf{A}^{N+1}\mathbf{C}$ and the algebra is semisimple for all $N \geq 1$. The remaining case, $\mathbf{H}_0$, has no localized junctions.

- The $\mathbf{E}_N$ series. Here we have the two realizations $\mathbf{E}_N = \mathbf{A}^{N-1}\mathbf{BCC}$ or $\tilde{\mathbf{E}}_N = \mathbf{A}^N\mathbf{X}_{[2,-1]}\mathbf{C}$, which are equivalent for $N \geq 2$. Here $\mathbf{E}_5 (= \mathbf{D}_5)$, $\mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8$ and $\mathbf{E}_9 = \widehat{\mathbf{E}}_8$ can be isolated. On the other hand all these give semisimple algebras so (ii) is not satisfied.

- The $\widehat{\mathbf{E}}_N$ series. Once more we have the two realizations $\widehat{\mathbf{E}}_N = \mathbf{A}^{N-1}\mathbf{BCCX}_{[3,1]}$ $(N \geq 1)$ and $\widehat{\widehat{\mathbf{E}}}_N = \mathbf{A}^N\mathbf{X}_{[2,-1]}\mathbf{CX}_{[4,1]}$ $(N \geq 0)$, which are again equivalent for $N \geq 2$. All these configurations can be isolated at least for $N \leq 9$, and correspond to parabolic monodromies. For $N \geq 3$ one identifies semisimple affine algebras and thus no possibly stable non-BPS states. Both $\widehat{\mathbf{E}}_2$ and $\widehat{\widehat{\mathbf{E}}}_1$ carry affine u(1) factors and thus are candidates for having possibly stable non-BPS states.

In conclusion, the conditions of isolation, plus (i) and (ii) have restricted the list to the cases of $\mathbf{D}_1, \widehat{\widehat{\mathbf{E}}}_1,$ and $\widehat{\mathbf{E}}_2$. We will now examine these cases in detail and confirm that they have collections of possibly stable non-BPS states, and therefore some genuinely stable non-BPS states.

4.1 Case of $\mathbf{D}_1$

The configuration here is $\mathbf{ABC}$ and having mutually nonlocal branes the lattice of localized junctions is one dimensional. This case is identical to a D7-brane O7-plane system analyzed in ref. [1]. Using the conditions of charge conservation one readily finds that this lattice is spanned by the minimal proper junction $\mathbf{J} = 2\mathbf{a} - \mathbf{b} - \mathbf{c}$. Here we are following the convention of [10, 8] that a junction $\mathbf{x}_{[p,q]}$ denotes a (p, q) string departing from the $\mathbf{X}_{[p,q]}$ 7-brane and going to ∞. We can easily verify that $\mathbf{J}$ satisfies conditions (i) and (ii). Indeed, using the rule [17] that each elementary junction $\mathbf{x}_{[p,q]}$ has self-intersection -1, and that the intersection number of a junction $\mathbf{x}_{[p,q]}$ with another junction $\mathbf{x}_{[p',q']}$ to its right is $\frac{1}{2}(pq' - qp')$, we get $\mathbf{J}^2 = -4$, and therefore condition (i) is satisfied. (The junction $\mathbf{J}$ corresponds to two strings departing the $\mathbf{A}$ brane and meeting after going around the $\mathbf{B}$ and $\mathbf{C}$ branes. In this picture the self-intersection is manifestly (-4).) Since any localized junction must be a multiple of the minimal junction $\mathbf{J}$, any junction must satisfy condition (i) and therefore there are no BPS junctions in this configuration. As a consequence condition (ii) is also satisfied. The states $(n\mathbf{J})$, for $n \neq 0$ are all possibly stable non-BPS states. In the limit when the separations between the 7-branes are large compared to the string length scale, the mass of a string junction can be computed reliably by integrating the tension along the various segments of the junction. In this classical limit the minimal mass configuration in the class of $n\mathbf{J}$ corresponds to n copies of the minimal mass junction in the class of $\mathbf{J}$. Thus the mass of the former is approximately n times the mass of the latter. Thus the minimal charged states $\pm\mathbf{J}$ are genuinely stable non-BPS. These were identified in [1].

4.2 Case of $\hat{\hat{\mathbf{E}}}_1$

This is the brane configuration having the following four seven branes (ref. [8, eqn. (3.10)])

$$\hat{\hat{\mathbf{E}}}_1 = \mathbf{A}\mathbf{X}_{[2,-1]}\mathbf{C}\mathbf{X}_{[4,1]} = \mathbf{A}\hat{\hat{\mathbf{E}}}_0 = \tilde{\mathbf{E}}_1\mathbf{X}_{[4,1]}\,. \tag{4.1}$$

We have written it in two ways; as an enhancement of $\hat{\hat{\mathbf{E}}}_0$, and as an affinization of $\widetilde{\mathbf{E}}_1$. Given that we have four seven-branes we must have two junctions spanning the lattice of localized junctions. We claim that the following is a basis for localized junctions of $\hat{\hat{\mathbf{E}}}_1$:

$$\begin{aligned} \bar{\mathbf{J}} &= 3\mathbf{a} - \mathbf{x}_{[2,-1]} - \mathbf{c}\,, \qquad \bar{\mathbf{J}}^2 = -8 \\ \delta &= \mathbf{x}_{[2,-1]} + 2\mathbf{c} - \mathbf{x}_{[4,1]}\,, \qquad \delta^2 = 0\,, \quad \delta\cdot\bar{\mathbf{J}} = 0\,. \end{aligned} \tag{4.2}$$

Linear independence is manifest, $\bar{\mathbf{J}}$ is supported on the $\mathbf{A}$ brane while δ is not, δ is supported on the $\mathbf{X}_{[4,1]}$ brane, while $\bar{\mathbf{J}}$ is not. After imposing the constraint that no D- or fundamental string charge flows to infinity, any arbitrary junction $p\mathbf{a}+q\mathbf{x}_{[2,-1]}+r\mathbf{c}+s\mathbf{x}_{[4,1]}$, with integers p,q,r,s, can be expressed as $-(r+2s)\bar{\mathbf{J}} - s\delta$. This establishes that $(\bar{\mathbf{J}},\delta)$ form a basis for the localized junctions of $\hat{\hat{\mathbf{E}}}_1$.

The physical interpretation of these junctions can be found by considering the subconfigurations. Indeed, the localized junctions of $\widetilde{\mathbf{E}}_1 = \mathbf{A}\mathbf{X}_{[2,-1]}\mathbf{C}$ make a one-dimensional lattice spanned by $\bar{\mathbf{J}}$. On the other hand the junction δ can be presented as string loop of charge $(p,q) = (-1,0)$ surrounding the configuration $\hat{\hat{\mathbf{E}}}_0$ (ref. [8, eqn. (3.11)]). This picture makes it manifest that $\delta^2 = 0$, a fact that guarantees that this junction arises from a holomorphic cycle of genus one and is therefore BPS [18]. Having charge $(-1,0)$ the junction δ can be moved across the remaining $\mathbf{A}$ brane and be presented as a loop surrounding the complete $\hat{\hat{\mathbf{E}}}_1$ configuration.[17] This makes $\bar{\mathbf{J}}\cdot\delta = 0$ manifest.

Since the arbitrary junction $\mathbf{J}_{Q,\ell} = Q\bar{\mathbf{J}} + \ell\delta$ satisfies $\mathbf{J}^2_{Q,\ell} = -8Q^2$, no junction with support on the $\mathbf{A}$ brane can be BPS. δ is the basis of BPS junctions. Thus the junctions $\mathbf{J}_{Q,\ell}$ with $Q \neq 0$ are all possibly stable non-BPS states. Among all such states there will be at least one lowest mass state $\mathbf{J}_{Q_0,\ell_0}$ that is a genuinely stable non-BPS state. The precise value of (Q_0,ℓ_0), however, is not determined by this argument. In fact, as we change the parameters labelling the isolated $\hat{\hat{\mathbf{E}}}_1$ configuration, the values of (Q_0,ℓ_0) can undergo discrete jumps. Note that for a fixed Q the non-BPS states $\mathbf{J}_{Q,\ell}$ for all values of ℓ generate a (level zero) representation of the $\widehat{\mathrm{u}(1)}$ algebra carried by the $\hat{\hat{\mathbf{E}}}_1$ configuration. Since this configuration is non-collapsible the affine symmetry is only spectrum generating, and states with different values of ℓ will typically have different masses and different stability properties.

[17]In fact such loop is the loop of the $\hat{\hat{\mathbf{E}}}_N$ configuration for all N.

JHEP03(2000)036

4.3 Case of $\widehat{\mathbf{E}}_2$

This is the configuration described in ref. [8, eqn. (3.7)]:

$$\widehat{\mathbf{E}}_2 = \mathbf{ABCCX}_{[3,1]} = \mathbf{E}_2\mathbf{X}_{[3,1]} = \mathbf{ABCBC} \tag{4.3}$$

expressed also as the enhancement of $\mathbf{E}_2$. Given that we have 5 seven-branes we expect three junctions to span the lattice of localized junctions. We claim that the basis of three junctions can be chosen to be

$$\begin{aligned} \mathbf{j} &= \mathbf{c}_1 - \mathbf{c}_2\,, \\ \mathbf{J}_- &= 2\mathbf{a} - \mathbf{b} - \mathbf{c}_1\,, \\ \delta' &= \mathbf{b} + \mathbf{c}_1 + \mathbf{c}_2 - \mathbf{x}_{[3,1]}\,, \end{aligned} \tag{4.4}$$

with

$$\mathbf{j}^2 = -2\,, \qquad \mathbf{J}_-^2 = -4\,, \qquad (\delta')^2 = 0\,, \qquad \mathbf{j}\cdot\mathbf{J}_- = 1\,, \qquad \mathbf{j}\cdot\delta' = \mathbf{J}_-\cdot\delta' = 0\,. \tag{4.5}$$

To see that this is a basis, we note that an arbitrary junction of the form $p\mathbf{a} + q\mathbf{b} + r\mathbf{c}_1 + s\mathbf{c}_2 + t\,\mathbf{x}_{[3,1]}$, with integers p, q, r, s, t, and satisfying the condition for D- and fundamental string charge localization, can be expressed as $-(r+s+2t)\mathbf{J}_- - (s+t)\mathbf{j} - t\delta'$.

The set of localized junctions of the $\mathbf{E}_2$ sub-configuration is spanned by $\mathbf{J}_-$ and $\mathbf{j}$. This carries an su(2) $\times$ u(1). $\mathbf{j}$ corresponds to the root of the su(2) factor and $\mathbf{J}_-$ is associated to the u(1) factor. We also introduce $\mathbf{J}_+ = 2\mathbf{a} - \mathbf{b} - \mathbf{c}_2$. The states $\mathbf{J}_\pm$ form a doublet of the su(2) satisfying

$$\mathbf{J}_\pm\cdot\mathbf{j} = \mp 1\,, \qquad \mathbf{J}_\pm^2 = -4\,. \tag{4.6}$$

The δ' junction is a $(-1,0)$ loop surrounding the configuration (ref. [8, eqn. (3.9)]). This explains why $\mathbf{j}\cdot\delta' = \mathbf{J}_-\cdot\delta' = 0$.

We now claim that no junction in $\widehat{\mathbf{E}}_2$ with support in the $\mathbf{A}$ brane can be BPS. Indeed, with

$$\mathbf{J}_{n,m,\ell} = n\mathbf{J}_- + m\mathbf{j} + \ell\delta' \;\rightarrow\; \mathbf{J}^2 = -4n^2 - 2m^2 + 2mn = -(m-n)^2 - 3n^2 - m^2\,. \tag{4.7}$$

Thus $\mathbf{J}^2 < -2$ for any non-zero integer n. Thus junctions in $\widehat{\mathbf{E}}_2$ with support on the $\mathbf{A}$ brane satisfy (i) and (ii), and are possibly stable non-BPS states. The u(1) charge is measured by the number of prongs on the $\mathbf{A}$ brane.

The su(2) symmetry is exact when the two $\mathbf{C}$ branes coincide. In this case $\mathbf{J}_\pm$ would make a doublet of possibly stable non-BPS states. The possibly non-BPS su(2) singlet of minimal u(1) charge is readily shown to be the junction $\mathbf{J}_0 = 2\mathbf{J}_- + \mathbf{j} + \ell\delta' = 4\mathbf{a} - 2\mathbf{b} - \mathbf{c}_1 - \mathbf{c}_2 + \ell\delta'$. It carries twice the u(1) charge of any member of the doublet. We can construct possibly non-BPS states in higher representations of $SU(2)$ in a similar manner. From the structure of the lattice it is easy to see that odd values of the u(1) charge must be associated to su(2) representations in the conjugacy of the doublet, while even u(1) charges must be associated to su(2) representations in the conjugacy class of the adjoint. Once we fix a u(1) and an su(2) representation, the states $\mathbf{J}_{n,m,\ell}$ for all values of ℓ generate a (level zero) representation of the affine $(\widehat{A}_1 \oplus \widehat{\mathrm{u}(1)}/\sim)$ algebra of the $\widehat{\mathbf{E}}_2$ configuration. Which of these configurations represent genuinely stable non-BPS state is a detailed dynamical question which we shall not address.

Finally we note that this configuration represents a single D7-brane near a pair of O7-planes. This is seen in the last presentation given in (4.3). The **A** brane represents a D7-brane, while each of the **BC** factors represents an O7-plane.

5. Non-BPS states on non-isolable configurations

The strategy that we have used so far in our search for stable non-BPS states consists of two steps. First we need a subset of 7-branes in F-theory on K3 such that there are non-BPS junctions living on this subset of branes which are stable against decay into other states living inside the same subsystem. Second, we need to ensure that these states are also stable against decay into junctions with one or more prongs on the 7-branes external to this subsystem.

The second condition requires that this subset of branes can be isolated and was the subject of study in sections 2 and 3. In this section we shall search for 7-brane configurations which satisfy the first condition and not the second. This would ensure that the non-BPS states living on this subsystem are stable against decay into BPS states living on the same subsystem, but could be unstable against decay into junctions with prongs on the external 7-branes. At present the significance of such brane configurations is not totally clear. However these configurations could be the starting point in our search for 7-brane configurations which admit non-BPS states which are stable due to dynamical reasons, namely that their mass is smaller (but not much smaller) than the possible decay products.

We shall begin by discussing two examples which we already encountered in section 4. The first example will be that of an $\widetilde{\mathbf{E}}_1$ configuration. This is generated by $\bar{\mathbf{J}}$ defined in eq. (4.2). Since $\bar{\mathbf{J}}^2 = -8$, any state of the form $n\mathbf{J}$ is non-BPS. The minimum mass state in this family will be stable against decay into other states living solely inside the $\widetilde{\mathbf{E}}_1$ brane system. When the relative separation between the branes is large, this corresponds to the junction $\pm\bar{\mathbf{J}}$. From [10] we know that for $\widetilde{\mathbf{E}}_1$, $\mathrm{Tr}(K) = -6$. Since the monodromy is hyperbolic, the $\widetilde{\mathbf{E}}_1$ configuration cannot be isolated.

The second example is that of $\mathbf{E}_2$. This is generated by the junctions $\mathbf{J}_-$ and $\mathbf{j}$ defined in eq. (4.4). Any junction of the form $\mathbf{J}_{m,n} = n\mathbf{J}_- + m\mathbf{j}$ has $\mathbf{J}^2 = -(m-n)^2 - 3n^2 - m^2 < -2$ for $n \neq 0$. Thus there should be at least one non-BPS state on this system with component along $\mathbf{J}_-$ which is stable against decay into other states living solely inside $\mathbf{E}_2$. Again we see from ref. [10] that in this case $\mathrm{Tr}(K) = -5$, and hence this configuration cannot be isolated.

We shall consider two more examples. The first example will be an arbitrary configuration of three seven branes. The second example will be that of a four 7-brane configuration carrying a $\mathrm{u}(1) \times \mathrm{u}(1)$ algebra.

Non-BPS states on three 7-branes. By an $\mathrm{SL}(2,\mathbb{Z})$ transformation, any three 7-brane configuration can be put in the form $\mathbf{AXX}'$ where $\mathbf{X}$ is a $[p,q]$ brane, and $\mathbf{X}'$ is a $[p',q']$ brane. We also require $q \neq 0$ and $q' \neq 0$, as well as $[p,q] \neq [p',q']$, for otherwise we have at least two mutually local branes and there will be BPS states carrying this U(1) charge into which a possible non-BPS state can decay. Without loss of generality we can also assume that both q and q' are positive. Define

$$\Delta = pq' - qp' \, . \tag{5.1}$$

The general junction is

$$\mathbf{J} = Q_A \mathbf{a} + Q\mathbf{x} + Q'\mathbf{x}' \, , \tag{5.2}$$

and using charge conservation to solve for Q_A and Q' in terms of Q, we find

$$\mathbf{J} = -\frac{Q\Delta}{q'}\,\mathbf{a} + Q\,\mathbf{x} - \frac{Qq}{q'}\,\mathbf{x}' \, . \tag{5.3}$$

Since $Q\Delta/q'$ and Qq/q' could be fractional, this junction is not necessarily proper. To address this issue, let us define $\ell = \gcd(q,q')$ and let $q' = \ell q_0$. One must then choose $Q = q_0$ to get the minimal proper junction. Indeed, this gives $Q/q' = 1/\ell$ and we get

$$\mathbf{J} = \frac{1}{\ell}\Big(-\Delta\,\mathbf{a} + q'\mathbf{x} - q\,\mathbf{x}'\Big) \, . \tag{5.4}$$

The self-intersection is readily found to be

$$\mathbf{J}^2 = -\frac{1}{\ell^2}\Big(\Delta^2 + q'^2 + q^2 + qq'\Delta\Big) \, , \qquad \ell = \gcd(q,q') \, , \qquad \Delta = pq' - qp' \, . \tag{5.5}$$

Each of the terms contributing to $\mathbf{J}^2$ is now an integer. We can easily choose p, q, p', q' such that $\mathbf{J}^2$ given above is < -2, so that all charged states living on this brane system are non-BPS states. The lightest of them will be stable against decay into other states within this system.

It is useful to write this in a more symmetric form. Let $\mathbf{z}_i$ denote as usual the elementary junction joining the i-th 7-brane to ∞. Using the notation of [10, section 2.1], we define $\mathbf{z}_{ij} = \mathbf{z}_i \times \mathbf{z}_j = (p_i q_j - q_i p_j)$, where (p_i, q_i) denotes the i-th 7-brane. Thus we have, in the present case, $\mathbf{z}_{12} = q$, $\mathbf{z}_{23} = \Delta$, $\mathbf{z}_{31} = -q'$, and moreover $\ell = \gcd(\mathbf{z}_{12}, \mathbf{z}_{23}, \mathbf{z}_{31})$. We thus have

$$\mathbf{J}^2 = -\frac{1}{\ell^2}\Big(\mathbf{z}_{12}^2 + \mathbf{z}_{23}^2 + \mathbf{z}_{31}^2 - \mathbf{z}_{12}\mathbf{z}_{23}\mathbf{z}_{31}\Big) \, . \tag{5.6}$$

JHEP03(2000)036

Ref. [10, eqn. (2.7)] for three 7-branes gives:

$$\mathrm{Tr}\, K = 2 - \mathbf{z}_{12}^2 - \mathbf{z}_{23}^2 - \mathbf{z}_{31}^2 + \mathbf{z}_{12}\mathbf{z}_{23}\mathbf{z}_{31}\,. \tag{5.7}$$

From this we see that

$$\mathbf{J}^2 = \frac{1}{\ell^2}\Big(\mathrm{Tr}\, K - 2\Big)\,. \tag{5.8}$$

Since in order to get a non-BPS junction we need $\mathbf{J}^2 < -2$, we must have

$$\mathrm{Tr}\, K < 2(1-\ell^2)\,. \tag{5.9}$$

Thus we see that except when $\ell = 1$, all three 7-brane configurations with stable non-BPS states will have $\mathrm{Tr}\, K < -2$, and thus have negative hyperbolic monodromies. When $\ell = 1$, one must have $\mathrm{Tr}\, K < 0$. As table 5 of [8] indicates, there is no three 7-brane configuration with $\mathrm{Tr}\, K = -1$, — a fact that is not hard to prove. The isolable configuration $\mathbf{D}_1$ corresponds to the case $\mathrm{Tr}\, K = -2$. The non-isolable three 7-brane configuration $\tilde{\mathbf{E}}_1$ corresponds to the case $\mathrm{Tr}\, K = -6$.

A case with $\mathrm{u}(1) \times \mathrm{u}(1)$**.** While refs. [10, 8] mostly searched for configurations with large symmetry algebras, it is clearly possible to put together many branes and still fail to find any enlarged semi-simple algebra. In such cases we must get u(1) factors. We illustrate this by considering a brane configuration with four 7-branes, which has no charged BPS states. The brane configuration is

$$\mathbf{X}_{[1,2]}\,\mathbf{ABC}\,, \tag{5.10}$$

which is obtained by adding the $\mathbf{X}_{[1,2]}$ brane to the $\mathbf{D}_1$ configuration. Since the lattice of localized junctions should be two-dimensional we expect to find two u(1)'s. One can show that the following is a basis for localized junctions:

$$\begin{aligned} \mathbf{J}_1 &= -2\mathbf{a} + \mathbf{b} + \mathbf{c} \\ \mathbf{J}_2 &= -\mathbf{a} + \mathbf{b} - \mathbf{c} + \mathbf{x}\,. \end{aligned} \tag{5.11}$$

These satisfy

$$\mathbf{J}_1^2 = -4\,, \qquad \mathbf{J}_2^2 = -4\,, \qquad \mathbf{J}_1 \cdot \mathbf{J}_2 = 0\,, \tag{5.12}$$

and therefore there are no BPS states on this brane configuration carrying either of these U(1) charges. This shows that this brane configuration has non-BPS junctions which are stable against decay within this brane system. The general localized junction on this brane configuration would be $\mathbf{J} = Q_1\mathbf{J}_1 + Q_2\mathbf{J}_2$, where Q_1 and Q_2 are the two u(1) charges of the junction. For this configuration $\mathrm{Tr}\, K = -14$, confirming that it cannot be isolated.

Acknowledgments

A.S. would like to thank the Center for Theoretical Physics at MIT for hospitality during part of this work, and the participants of the ICTP workshop on String Theory for providing a stimulating environment during the course of this work. B.Z. would like to thank the Department of Physics at Harvard University, for hospitality during the concluding stage of this work.

The work of B.Z. is supported by the U.S. Department of Energy under contract # DE-FC02-94ER40818.

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